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DAMPING CAPACITY OF SOIL DURING DYNAMIC LOADING REPORT  
I. REVIEW OF MATHEMATICAL MATERIAL MODELS

Behzad Rohani

Army Engineer Waterways Experiment Station  
Vicksburg, Mississippi

April 1972

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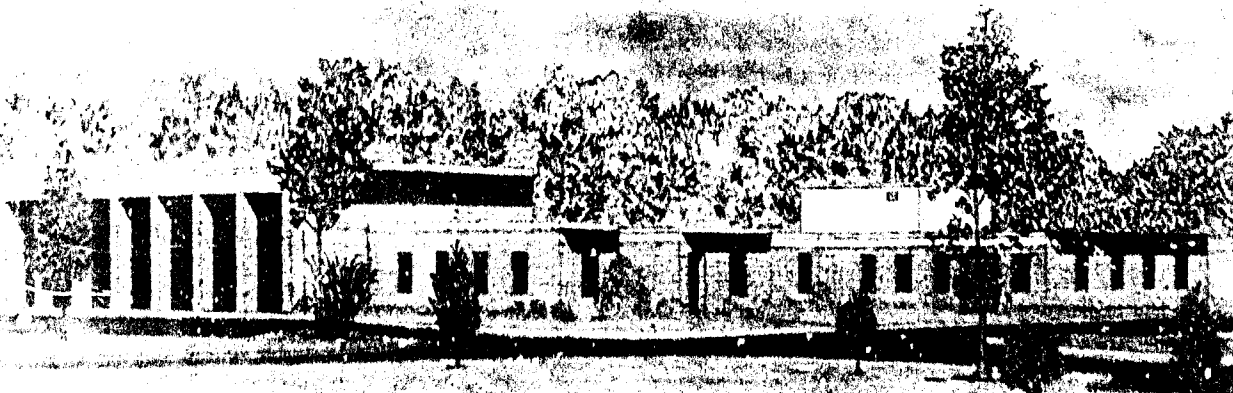
# DAMPING CAPACITY OF SOIL DURING DYNAMIC LOADING

Report I

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by

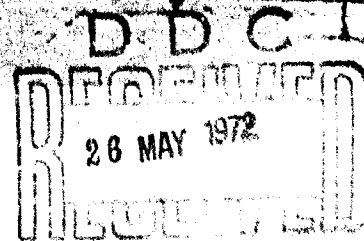
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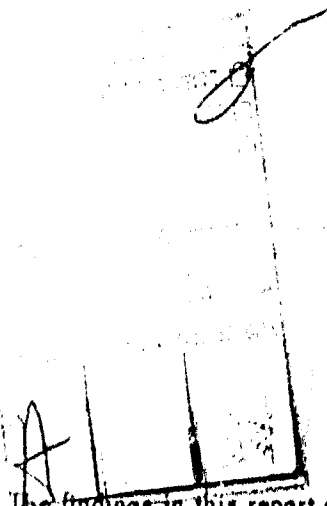
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Conducted by U. S. Army Engineer Waterways Experiment Station, Vicksburg, Mississippi

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## FOREWORD

This investigation was conducted by the U. S. Army Engineer Waterways Experiment Station (WES) under the sponsorship of the Office, Chief of Engineers, Department of the Army, as a part of Project 4A061102B52E, "Evaluation of the Damping Capacity of Soils Under Dynamic Loads," Task 01, Work Unit 012.

The investigation was conducted by Drs. B. Rohani and G. Y. Baladi during the period April-June 1971 under the general direction of Messrs. J. P. Sale, Chief, R. G. Ahlvin, R. W. Cunny, and Dr. L. W. Heller of the Soils Division. The report was written by Dr. Rohani who wishes to acknowledge the assistance and suggestions of Dr. Baladi.

Director of WES during the preparation and publication of this report was COL Ernest D. Peixotto, CE. Technical Director was Mr. F. K. Brown.

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## SUMMARY

To establish basic definitions and to serve as a reference for future theoretical and applied research efforts, the fundamental concepts of the theory of continuous mass media are reviewed, and the mathematical foundations of constitutive equations for isotropic and homogeneous materials are presented. The results are restricted to deformation for which isothermal behavior prevails and displacement gradients are small. Seven types of mathematical material models are discussed that can be used to describe the stress-strain-time behavior of physically nonlinear materials. The models are given in three-dimensional tensorial representation.

The energy-dissipation properties of engineering materials are discussed within the framework of the above-mentioned mathematical material models, and the damping models used in engineering analyses of vibratory systems are summarized.

## DAMPING CAPACITY OF SOIL DURING DYNAMIC LOADING

### REVIEW OF MATHEMATICAL MATERIAL MODELS

#### PART I: INTRODUCTION

##### Background

1. A great number of engineering materials exhibit hysteretic effects under cyclic or near-cyclic loading conditions, even when undergoing small deformation. A prime example is soil. The energy-dissipation characteristics of soil can be observed, for example, from the attenuation of the stress pulse during one-dimensional wave propagation tests or from the formation of a hysteretic loop during a load-unload cycle of deformation. The hysteretic character of the stress-strain behavior of soil plays an important role in the determination of the response of soil to dynamic loads and must be incorporated in the formulation and solution of soil dynamics boundary-value problems.

2. The term damping is often used, in a general sense, to describe the energy-dissipation properties of engineering materials; and various mathematical material models have been devised to represent such phenomena. The approach used to formulate most of these models is based on the methods of continuum mechanics, which disregard the atomistic details of the material and consider its gross behavior only. The models are, therefore, phenomenological in nature. In fact, no satisfactory explanation of damping mechanism on the atomistic level is available at the present time, although several hypotheses have been set forth in recent years. For the solution of many engineering problems, however, a mathematical model of damping that is based on the concepts of continuum mechanics is quite sufficient and useful. Such a model must, of course, be general enough to describe the energy-dissipation properties of the material under various states of stress and deformation.

3. To date, two alternate procedures have been employed to

describe the hysteretic behavior or the energy-dissipation properties of soil. Both procedures are based on the methods of continuum mechanics. In the first procedure, the hysteretic behavior of soil is accounted for by using three sets of time-independent stress-strain relations: one set for initial loading, one for unloading, and one for subsequent reloading. The mathematical expressions for the stress-strain relations are usually derived from cyclic or near-cyclic experimental data by curve-fitting techniques. A set of criteria or logics is also specified to determine whether the material under consideration is being loaded, unloaded, or reloaded so that the proper stress-strain relations can be used. The theory of plasticity is the outcome of this procedure. In the second procedure, a time-dependent mathematical model is postulated, and the hysteretic behavior of soil is accounted for by the appearance of the time derivatives (deformation rate) in the model. The theory of viscoelasticity is the outcome of this procedure.

4. Both of these techniques have been used extensively in the formulation and solution of soil dynamics boundary-value problems. The mechanism of energy dissipation in the first procedure, however, is different from that in the second procedure. At the present time, it is not clear which procedure is the more correct one, and the available experimental information on the stress-strain behavior of soil under dynamic cyclic loading is not conclusive enough to favor one or the other. For reasons of practicability or mathematical simplicity, and depending on the type of problem being considered, some investigators use the first procedure while others use the second. A subsequent report (Report 2) will review current laboratory methods of determining the damping capacity of soil; it will describe specific techniques used to evaluate hysteretic effects within the framework of both the time-independent and the time-dependent mathematical material models.

#### Purpose

5. The purpose of this report is to review and document various mathematical material models that can be used to describe the

stress-strain-time behavior of physically nonlinear, isotropic, and homogeneous materials. Emphasis is placed on the material models that exhibit hysteretic effects under cyclic loading. These models will serve as bases for future work for the evaluation of damping capacity of soils under dynamic loads.

#### Scope

C. Since the mathematical material models presented in this report are based on the methods of continuum mechanics, a brief discussion of the fundamental concepts of the theory of continuous media is given in Part II. Part III contains a summary of the available mathematical material models in their general functional representation. In Part IV, various types of damping models commonly used in engineering analyses of vibratory systems are presented.

## PART II: CONCEPTS FROM CONTINUUM MECHANICS

### Basic Principles

7. In engineering practice it is convenient, and often reasonable, to disregard the atomistic details of matter and consider its gross behavior only. It is also customary to adopt the hypothesis that matter can be replaced by a mathematical model whose kinematic or dynamic variables are piecewise continuous functions of spatial coordinates and time. Such a medium is often referred to as a continuum. The motion of any continuum in a Galilean frame of reference is governed by the following laws:

- a. Conservation of mass
- b. Conservation of energy
- c. Balance of linear momentum
- d. Balance of angular momentum
- e. Principle of inadmissibility of decreasing entropy

These laws constitute the basic axioms of continuum mechanics.<sup>1</sup> If mechanical energy is the only form of energy to be considered in a problem, the above principles will lead to the continuity equation\*

$$\frac{\partial \rho}{\partial t} + (\rho v_i)_{,i} = 0 \quad (1)$$

and the equations of motion

$$\sigma_{ij,j} + \rho(F_i - a_i) = 0 \quad (2)$$

where

$\rho$  = mass density

$v_i$  = components of velocity vector

---

\* Indices take on values 1, 2, or 3. A repeated index is to be summed out over its range. Comma in the subscripts represents a derivative. Quantities are referred to rectangular Cartesian coordinates  $X_i$ .

$\sigma_{ij}$  = symmetrical stress tensor

$F_i$  = components of body force

$a_i$  = components of acceleration vector

8. Equations 1 and 2 constitute four equations that involve ten unknown functions of time and space: the mass density  $\rho$ , the three velocity components  $v_i$ , and the six independent stress components  $\sigma_{ij}$ . The body force components  $F_i$  are known quantities and the acceleration components  $a_i$  are expressible in terms of the velocity components  $v_i$ . Obviously equations 1 and 2 are inadequate to determine the motion or deformation of a medium when subjected to external disturbances such as surface forces and/or displacements. Therefore, six additional equations relating the ten unknown variables  $\rho$ ,  $v_i$ , and  $\sigma_{ij}$  are required. In the field of continuum mechanics, such relations are stated by constitutive equations, which relate the stress tensor  $\sigma_{ij}$  to the deformation and time rate of deformation. The difference between the constitutive equations and the field equations (equations 1 and 2) is that the latter are applicable to all materials, whereas the former represent the intrinsic response of a particular material. Furthermore, a constitutive equation provides a mathematical description or definition of an ideal material rather than a statement of a universal law.

§. The general form of a constitutive equation for isothermal conditions may be expressed by the functional form

$$\epsilon_{mn}(\rho, v_k, \sigma_{ij}) = 0 \quad (3)$$

relating the ten unknown variables  $\rho$ ,  $v_k$ , and  $\sigma_{ij}$ . Equation 3 can be written in a more definite form relating the stress tensor to the strain tensor and the deformation-rate tensor, i.e.,

$$\sigma_{ij} = f_{ij}(\epsilon_{mn}, d_{rs}) \quad (4)$$

where

$\epsilon_{mn}$  = strain tensor

$d_{rs}$  = deformation-rate tensor

If the displacement gradients are small, the strain tensor can be expressed in terms of the components of the displacement vector  $u_i$  by the following relation

$$\epsilon_{mn} = \frac{1}{2} (u_{m,n} + u_{n,m}) \quad (5)$$

The deformation-rate tensor is related to the components of the velocity vector  $v_i$ , i.e.

$$d_{rs} = \frac{1}{2} (v_{r,s} + v_{s,r}) \quad (6)$$

The mass density  $\rho$  can be related to  $d_{rs}$  or  $\epsilon_{mn}$  through equations 1, 5, and 6. Equations 1, 2, and 4, therefore, constitute ten equations in ten unknowns and will lead, in conjunction with the kinematic relations given by equations 5 and 6, to a complete description of the boundary-value problem. In addition to the above-mentioned equations, boundary conditions in terms of boundary displacements and/or surface tractions must also be specified to define the particular problem of interest.

#### Requirements for Constitutive Equations

10. In order for the constitutive equation (equation 4) to describe a physical material adequately, the response function  $f_{ij}$  must be form invariant with respect to rigid motion of spatial coordinates. This requirement stems from the fact that the response of a material is independent of the motion of the observer. Furthermore,  $f_{ij}$  must be expressed in tensor language to ensure that the constitutive equation is invariant to coordinate transformations. In addition to the above-mentioned invariance principles, the response function  $f_{ij}$  must be consistent with the general principles of conservation or balance of mass, momentum, and energy. If it is assumed that the material under consideration is isotropic and elastic (linear or nonlinear), every principal axis of strain must also be a principal axis of stress, i.e.,

if  $\epsilon_{12} = \epsilon_{23} = \epsilon_{31} = 0$ ,  $f_{ij}$  must show that  $\sigma_{12} = \sigma_{23} = \sigma_{31} = 0$ . Also, to avoid stress-induced anisotropy for an isotropic elastic medium,  $f_{ij}$  must ensure that a zero state of strain corresponds to a zero or a specified scalar state of stress.

### The General Form of Isotropic Constitutive Equation

11. The general form of the response function  $f_{ij}$  satisfying the invariance principles and the requirements of isotropy was derived by Rivlin and Ericksen.<sup>2</sup> Functions satisfying these conditions are called hemitropic functions of their arguments. A hemitropic polynomial  $F$  of two symmetric second-rank tensor variables  $A$  and  $B$  admits a representation of the form

$$F(A,B) = \phi_0 I + \phi_1 A + \phi_2 A^2 + \phi_3 B + \phi_4 B^2 + \phi_5 (AB + BA) + \phi_6 (A^2 B + BA^2) \\ + \phi_7 (AB^2 + B^2 A) + \phi_8 (A^2 B^2 + B^2 A^2) \quad (7)$$

in which  $I = \delta_{ij}$  = the Kronecker delta ; and the coefficients  $\phi_0, \dots, \phi_8$  are scalar-valued functions of the ten joint invariants of  $A$  and  $B$ .<sup>3</sup> The ten joint invariants of  $A$  and  $B$  are given by

$$\begin{array}{lll} \text{tr}(A) & \text{tr}(A^2) & \text{tr}(A^3) \\ \text{tr}(B) & \text{tr}(B^2) & \text{tr}(B^3) \\ \text{tr}(AB) & \text{tr}(A^2 B) & \text{tr}(B^2 A) \\ & \text{tr}(A^2 B^2) & \end{array} \quad (8)$$

where  $\text{tr}$  = trace of , indicating the sum of the diagonal terms of a square matrix.

12. According to equation 7, the admissible form of the isothermal constitutive equation (equation 4), or  $f_{ij}$ , takes the following form



$$\begin{aligned}
\sigma_{ij} = & \eta_0 \delta_{ij} + \eta_1 \epsilon_{ij} + \eta_2 \epsilon_{im} \epsilon_{mj} + \eta_3 d_{ij} + \eta_4 d_{im} d_{mj} + \eta_5 (\epsilon_{im} d_{mj} + d_{im} \epsilon_{mj}) \\
& + \eta_6 (\epsilon_{im} \epsilon_{mn} d_{nj} + d_{im} \epsilon_{mn} \epsilon_{nj}) + \eta_7 (\epsilon_{im} d_{mn} d_{nj} + d_{im} d_{mn} \epsilon_{nj}) \\
& + \eta_8 (\epsilon_{im} \epsilon_{mn} d_{np} d_{pj} + d_{im} d_{mn} \epsilon_{np} \epsilon_{pj})
\end{aligned} \tag{9}$$

The response coefficients  $\eta_0, \dots, \eta_8$  are scalar-valued functions of the ten joint invariants of the strain tensor and the deformation-rate tensor. In view of equations 5, 6, and 8, the ten joint invariants of the strain tensor and the deformation-rate tensor take the following representation:

$$\begin{aligned}
I_1 &= \epsilon_{ss} & \dot{I}_1 &= d_{ss} \\
I_2 &= \epsilon_{ts} \epsilon_{st} & \dot{I}_2 &= d_{ts} d_{st} \\
I_3 &= \epsilon_{ts} \epsilon_{sr} \epsilon_{rt} & \dot{I}_3 &= d_{ts} d_{sr} d_{rt} \\
M &= \epsilon_{ts} d_{st} & R &= \epsilon_{ts} d_{sr} d_{rt} \\
N &= \epsilon_{ts} \epsilon_{sr} d_{rt} & Q &= \epsilon_{ts} \epsilon_{sk} d_{kr} d_{rt}
\end{aligned} \tag{10}$$

These response coefficients take various forms for different materials and must be determined from experimental observation. There is, however, no a priori reason for requiring that all response coefficients appear in the constitutive equations for all materials. Some of the response coefficients may vanish for some materials. Accordingly, a constitutive equation with two response coefficients, such as Hooke's law, is as valid and significant as a constitutive equation that includes the nine possible response coefficients  $\eta_0, \dots, \eta_8$ . The difference between the two equations is in their range of application.

13. It should be pointed out that equation 9 is not based on thermodynamic considerations and the response coefficients are not, in general, related to a single potential function. A special form of equation 9 based on thermodynamic considerations was also developed

by Schapery<sup>4</sup> and has been used extensively for stress analysis of visco-elastic materials.

14. Presently there are several forms of isotropic constitutive equations available that can be used to describe the stress-strain-time response of various materials. They are all derived from equation 7 or 9 and are presented in the following part.

### PART III: REVIEW OF ISOTROPIC CONSTITUTIVE EQUATIONS

#### General

15. The theoretical foundation of isotropic constitutive equations is based on the Rivlin-Ericksen equation presented in Part II (equation 7). Various classes of constitutive equations have been developed in recent years utilizing equation 7 or 9. A summary of these equations is presented in the following sections.

#### Elastic Materials

16. For elastic materials, the state of stress is a function of the current state of strain, i.e.

$$\sigma_{ij} = f_{ij}(\epsilon_{mn}) \quad (11)$$

The response function  $f_{ij}$  has the same form as equation 7 where  $A = \epsilon_{mn}$  and  $B = 0$ . Accordingly, the constitutive equation for elastic materials becomes

$$\sigma_{ij} = \phi_0 \delta_{ij} + \phi_1 \epsilon_{ij} + \phi_2 \epsilon_{im} \epsilon_{mj} \quad (12)$$

Equation 12 is often referred to as the Cauchy elastic constitutive equation.<sup>1</sup> The response coefficients  $\phi_0$ ,  $\phi_1$ , and  $\phi_2$  are functions of the three strain invariants  $I_1$ ,  $I_2$ , and  $I_3$  given by equation 10. Equation 12 can describe the mechanical behavior of various types of elastic materials by properly selecting the forms of  $\phi_0$ ,  $\phi_1$ , and  $\phi_2$ . For example, if  $\phi_2 = 0$ ,  $\phi_1 = 2\mu$ , and  $\phi_0 = \lambda I_1$ , equation 12 degenerates to Hooke's law, where  $\mu$  and  $\lambda$  are the Lamé constants. For a second-order elastic stress-strain relation, the response coefficients take the forms  $\phi_2 = D$ ,  $\phi_1 = 2\mu + CI_1$ , and  $\phi_0 = \lambda I_1 + AI_1^2 + BI_2$  where  $A$ ,  $B$ ,  $C$ , and  $D$  are material constants. Therefore, a second-order elastic stress-strain law, based on

equation 12, involves six material constants. Paralleling the above procedure, higher order and more complicated elastic stress-strain relations can be developed.

17. The counterpart of equation 12, which is derived from the conservation of energy and is referred to as the hyperelastic constitutive equation, is given as

$$\sigma_{ij} = \frac{\partial U}{\partial I_1} \delta_{ij} + 2 \frac{\partial U}{\partial I_2} \epsilon_{ij} + 3 \frac{\partial U}{\partial I_3} \epsilon_{im} \epsilon_{mj} \quad (13)$$

where  $U = U(I_1, I_2, I_3)$  is the strain-energy density function.<sup>5</sup> Comparison of equation 13 with equation 12 indicates that hyperelastic materials are special forms of the Cauchy elastic materials, where

$$\phi_0 = \frac{\partial U}{\partial I_1} ; \phi_1 = 2 \frac{\partial U}{\partial I_2} ; \phi_2 = 3 \frac{\partial U}{\partial I_3} \quad (14)$$

Consequently, a Cauchy elastic material is hyperelastic if the response coefficients  $\phi_0$ ,  $\phi_1$ , and  $\phi_2$  are related in the following manner

$$\frac{\partial \phi_1}{\partial I_1} = 2 \frac{\partial \phi_0}{\partial I_2} \quad (15a)$$

$$\frac{\partial \phi_1}{\partial I_3} = \frac{2}{3} \frac{\partial \phi_2}{\partial I_2} \quad (15b)$$

$$\frac{\partial \phi_2}{\partial I_1} = 3 \frac{\partial \phi_0}{\partial I_3} \quad (15c)$$

For linear elastic materials

$$U = \frac{\lambda}{2} I_1^2 + \mu I_2 \quad (16)$$

and both equations 12 and 13 yield the same results. The effect of the thermodynamic restrictions, therefore, is not evident when considering linear elastic materials. However, the effect of thermodynamic

restrictions becomes very pronounced when considering higher order stress-strain laws. For example, a second-order elastic stress-strain law formulated from the Cauchy elastic constitutive equation involves, as was shown previously, six material constants. On the other hand, a second-order elastic stress-strain law based on the hyperelastic constitutive equation (equation 13) involves only five material constants,<sup>6</sup> since the strain energy function for second-order hyperelastic materials takes the form  $U = \frac{\lambda}{2} I_1^2 + \mu I_2 + A_1 I_1^3 + B_1 I_1 I_2 + C_1 I_3$  where  $A_1$ ,  $B_1$ , and  $C_1$  are material constants. The effect of thermodynamic restrictions, in this case, is to reduce the number of material constants from six to five. However, the physical implications of this reduction in material constants is not clear at the present time.

18. The inverse forms of the Cauchy elastic and hyperelastic constitutive equations, resulting in strain-stress laws, are given as

$$\epsilon_{ij} = \theta_0 \delta_{ij} + \theta_1 \sigma_{ij} + \theta_2 \sigma_{im} \sigma_{mj} \quad (17)$$

for the Cauchy elastic materials, and

$$\epsilon_{ij} = \frac{\partial \psi}{\partial J_1} \delta_{ij} + 2 \frac{\partial \psi}{\partial J_2} \sigma_{ij} + 3 \frac{\partial \psi}{\partial J_3} \sigma_{im} \sigma_{mj} \quad (18)$$

for hyperelastic materials. The response coefficients  $\theta_0$ ,  $\theta_1$ , and  $\theta_2$ , as well as the complementary-energy density function  $\psi$ , are functions of the following stress invariants

$$\begin{aligned} J_1 &= \sigma_{ss} \\ J_2 &= \sigma_{ts} \sigma_{st} \\ J_3 &= \sigma_{ts} \sigma_{sr} \sigma_{rt} \end{aligned} \quad (19)$$

For linear elastic materials, the complementary-energy density function is given as

$$\psi = \frac{1+\nu}{2E} J_2 - \frac{\nu}{2E} J_1^2 \quad (20)$$

where  $E$  and  $\nu$  are Young's modulus of elasticity and Poisson's ratio, respectively. If equation 17 is utilized to derive the strain-stress law for linear elastic materials, the response coefficients take the forms  $\theta_2 = 0$ ,  $\theta_1 = \frac{1+\nu}{E}$ , and  $\theta_0 = -\frac{\nu}{E} J_1$ . Again, as expected, both the Cauchy elastic and hyperelastic constitutive equations yield the same strain-stress laws for linear elastic materials.

### Hypoelastic Materials

19. The theory of hypoelasticity was formulated by Truesdell.<sup>7</sup> The theory is formulated in terms of rates of both stress and deformation and is intended to describe the mechanical behavior of path-dependent materials. As was pointed out in the previous section, the state of stress for an elastic material is a function of the current state of strain and is independent of the path followed to reach that state. For real materials, earth materials in particular, the final state of stress is a function of the final state of strain as well as the stress path used to reach the final state. Hypoelastic theory predicts this type of behavior and may be used to model the stress-strain behavior of soil.

20. The basic constitutive equation of hypoelastic materials is expressed in the form

$$\hat{s} = \bar{f}(d, \bar{s}) \quad (21)$$

where

$$\hat{s} = \text{nondimensional stress flux tensor} = \frac{\hat{\sigma}_{ij}}{2\mu}$$

$$\bar{s} = \text{nondimensional stress tensor} = \frac{\sigma_{ij}}{2\mu}$$

$$\hat{\sigma}_{ij} = \text{stress flux tensor}$$

$$d = \text{deformation-rate tensor (equation 6)}$$

Jaumann's form of stress flux tensor is given by<sup>1</sup>

$$\hat{\sigma}_{ij} = \frac{\partial \sigma_{ij}}{\partial t} + \sigma_{ij,k} v_k + \sigma_{ik} \omega_{kj} - \sigma_{kj} \omega_{ik} \quad (22)$$

where

$$\omega_{rs} = \frac{1}{2} (v_{r,s} - v_{s,r}) \quad (23)$$

is the Eulerian spin tensor. This definition of stress flux is not unique, and various other forms developed by Truesdell and Oldroyd are available,<sup>1</sup> which differ from Jaumann's form in the terms containing the spin tensor. However, at the present time there is no a priori reason for utilizing any specific form of the stress flux tensor; additional physical postulates, or experimental information, are required in order to determine a preferred form of this tensor. For static problems or for dynamic problems with nearly irrotational displacement fields, the stress-rate tensor (the first term on the right side of equation 22) may be used instead of the stress flux tensor.

21. The hypoelastic response function  $\bar{f}(d, \bar{s})$  is of the same form as equation 7 where  $A = d$  and  $B = \bar{s}$ . Accordingly, the hypoelastic constitutive equation (equation 21) becomes

$$\begin{aligned} \hat{s} = & \alpha_0 I + \alpha_1 d + \alpha_2 d^2 + \alpha_3 \bar{s} + \alpha_4 \bar{s}^2 + \alpha_5 (d\bar{s} + \bar{s}d) \\ & + \alpha_6 (d^2 \bar{s} + \bar{s}d^2) + \alpha_7 (d\bar{s}^2 + \bar{s}^2 d) + \alpha_8 (d^2 \bar{s}^2 + \bar{s}^2 d^2) \end{aligned} \quad (24)$$

where  $\alpha_0, \dots, \alpha_8$  are scalar-valued functions of the ten joint invariants of  $d$  and  $\bar{s}$  (joint invariants of  $d$  and  $\bar{s}$  are obtained from equation 8 by substituting  $d$  and  $\bar{s}$  for  $A$  and  $B$ , respectively). The response coefficients  $\alpha_0, \dots, \alpha_8$  are further restricted by the first hypothesis of hypoelasticity: "No constitutive coefficients of a hypoelastic material shall carry a dimension independent of the dimension of stress." The consequence of this hypothesis on the hypoelastic constitutive equation is that all terms containing second and higher powers of  $d$  must vanish. Thus, the response coefficients  $\alpha_8 = \alpha_6 = \alpha_2 = 0$ ;  $\alpha_7, \alpha_5$ , and  $\alpha_1$  must be independent of  $d$  and functions of  $\bar{s}$  alone; and  $\alpha_0, \alpha_3$ , and  $\alpha_4$  must be of degree one in  $d$ . Imposing the above restrictions on the response coefficients

in equation 24, the hypoelastic constitutive equation reduces to

$$\begin{aligned} \hat{s} = & \dot{I}_1 \beta_0 I + \beta_1 d + \dot{I}_1 \beta_2 \bar{s} + E \beta_3 I + \frac{\beta_4}{2} (d\bar{s} + \bar{s}d) + \dot{I}_1 \beta_5 \bar{s}^2 \\ & + E \beta_6 \bar{s} + F \beta_7 I + \frac{\beta_8}{2} (d\bar{s}^2 + \bar{s}^2 d) + E \beta_9 \bar{s}^2 + F \beta_{10} \bar{s} + F \beta_{11} \bar{s}^2 \end{aligned} \quad (25)$$

in which  $\beta_0, \dots, \beta_{11}$  are dimensionless functions of the three principal invariants of  $\bar{s}$  only, and  $E$  and  $F$  are defined as

$$\begin{aligned} E &= \bar{s}_{mr} d_{nm} \\ F &= \bar{s}_{mn} \bar{s}_{np} d_{pm} \end{aligned} \quad (26)$$

22. From equation 25, it is apparent that the constitutive equations of hypoelasticity are coupled differential equations of first order. Furthermore, the differential equations are homogeneous in time. To obtain a unique solution to these equations, some initial conditions that are consistent with the invariance principles must be prescribed. The integration of the differential equations, for a given stress path, leads to stress-strain relations. Thus, a relation between stress and strain is the outcome of the theory.

23. Truesdell<sup>7</sup> has defined various classes of hypoelastic materials that are characterized by the highest degree of  $\bar{s}$  appearing in equation 25. If the right side of equation 25 is independent of  $\bar{s}$ , the material is called hypoelastic material of grade zero. In this case equation 25 reduces to

$$\hat{s} = \dot{I}_1 \beta_0 I + \beta_1 d \quad (27)$$

which is directly similar to the constitutive equation for a linear, isotropic, elastic material if  $\beta_0 = \frac{\lambda}{2\mu}$  and  $\beta_1 = 1$ , i.e.

$$\hat{\sigma}_{ij} = \lambda d_{kk} \delta_{ij} + 2\mu d_{ij} \quad (28)$$



If the right side of equation 25 contains up to the first power of stress, the material is called hypoelastic of grade one and so on.

24. In establishing the constitutive equation of hypoelastic materials, no assumptions were made in regard to the magnitude of stresses and strains involved; hence, the theory is applicable for all motions.

### Hygrosteric Materials

25. The term hygrosteric is used for materials having a constitutive equation of the form

$$\hat{s} = \bar{f}(d, \bar{s}, \rho) \quad (29)$$

where the response function  $\bar{f}$  is a polynomial in  $d$  and  $\bar{s}$  with coefficients depending on the density  $\rho$ . Due to the scalar character of the density  $\rho$ , the formulation of constitutive equations for hygrosteric materials is the same as for the hypoelastic materials except that the response coefficients appearing in equation 25 may be considered functions of  $\rho$  or the dimensionless ratio  $\rho/\rho_0$  where  $\rho_0$  is the initial mass density of the material.

### Stokesian Fluids

26. Stokesian fluids are characterized by the constitutive equations of the following form

$$\sigma_{ij} = f_{ij}(d_{rs}) \quad (30)$$

The state of stress, therefore, is a function of the current rate of deformation. In view of equation 9, equation 30 becomes

$$\sigma_{ij} = \eta_0 \delta_{ij} + \eta_3 d_{ij} + \eta_4 d_{im} d_{mj} \quad (31)$$

where  $\eta_0$ ,  $\eta_3$ , and  $\eta_4$  are functions of the three deformation-rate invariants given by equation 10. Equation 31 can describe the behavior of various fluids by proper selection of the response coefficients  $\eta_0$ ,  $\eta_3$ , and  $\eta_4$ . For example, if  $\eta_4 = 0$ ,  $\eta_3 = 2\mu_v$ , and  $\eta_0 = -P + \lambda_v \dot{I}_1$ , where  $\mu_v$  and  $\lambda_v$  are the shear and dilatational viscosity coefficients, respectively, and  $P$  is pressure, equation 31 degenerates to the constitutive equation of linear viscous fluids, i.e.

$$\sigma_{ij} = -P\delta_{ij} + \lambda_v \dot{I}_1 \delta_{ij} + 2\mu_v d_{ij} \quad (32)$$

### Viscoelastic Materials

27. The constitutive equation of viscoelastic materials is expressed by

$$\sigma_{ij} = f_{ij}(\epsilon_{mn}, d_{rs}) \quad (33)$$

where the isotropic function  $f_{ij}$  is given by equation 9. Equation 33 reduces to the Cauchy elastic constitutive equation (equation 12) if dependence on  $d_{rs}$  disappears. Various classes of viscoelastic materials can be described by equation 33 (or equation 9) by proper selection of the response coefficients  $\eta_0, \dots, \eta_8$ . For example, if  $\eta_0 = \lambda I_1 + \lambda_v \dot{I}_1$ ,  $\eta_1 = 2\mu$ ,  $\eta_3 = 2\mu_v$ , and  $\eta_2 = \eta_4 = \eta_5 = \eta_6 = \eta_7 = \eta_8 = 0$ , equation 9 reduces to the constitutive equation of the Kelvin-Voigt material, i.e.

$$\sigma_{ij} = \lambda I_1 \delta_{ij} + \lambda_v \dot{I}_1 \delta_{ij} + 2\mu \epsilon_{ij} + 2\mu_v d_{ij} \quad (34)$$

Equation 34 is the three-dimensional representation of the parallel spring-dashpot model used in the theory of vibration. Taking  $\eta_5 = \eta_6 = \eta_7 = \eta_8 = 0$  in equation 9, a second-order viscoelastic constitutive model, often referred to as the nonlinear Kelvin solid, can result, i.e.

$$\sigma_{ij} = \eta_0 \delta_{ij} + \eta_1 \epsilon_{ij} + \eta_2 \epsilon_{im} \epsilon_{mj} + \eta_3 d_{ij} + \eta_4 d_{im} d_{mj} \quad (35)$$

Equation 35 includes tensorial as well as scalar nonlinearities and can no longer be presented by a simple parallel spring and dashpot model. Many complex aspects of the physical behavior of real materials, such as shear-dilatancy phenomenon and the Poynting effect,<sup>1</sup> can be described by equation 35.

28. Viscoelastic constitutive equations can also be formulated in integral forms, e.g.

$$\sigma_{ij} = \int_0^t \lambda(t - \tau) \frac{dI_1}{d\tau} \delta_{ij} d\tau + 2 \int_0^t \mu(t - \tau) \frac{d\epsilon_{ij}}{d\tau} d\tau \quad (36)$$

where  $\lambda$  and  $\mu$  are referred to as relaxation moduli or memory functions and  $(t - \tau)$  is an elapsed time. Other forms of viscoelastic constitutive equations, e.g., series forms with differential operators as arguments, are also available that can be used to characterize various viscoelastic materials.<sup>1</sup> Viscoelastic constitutive equations that are expressed in the form of equation 9 (e.g., equations 34 and 35), however, possess a certain mathematical simplicity that makes them more attractive for engineering analyses than other forms of viscoelastic constitutive equations.

### Viscoplastic Materials

29. To describe the mechanical behavior of rate-dependent materials that are compactible, i.e. exhibit time-independent as well as time-dependent hysteretic effects, constitutive equations of the following form are often used:

$$\sigma_{ij} = f_{ij}(\dot{\sigma}_{pq}, \epsilon_{mn}, d_{rs}) \quad (37)$$

where  $\dot{\sigma}_{pq}$  are the components of the stress-rate tensor. The isotropic response function  $f_{ij}$  is a hemitropic function of  $\dot{\sigma}_{pq}$ ,  $\epsilon_{mn}$ , and  $d_{rs}$  and takes the following representation:<sup>8</sup>

$$\begin{aligned}
\sigma_{ij} = & A_0 \delta_{ij} + A_1 \dot{\sigma}_{ij} + A_2 \dot{\sigma}_{im} \dot{\sigma}_{mj} + A_3 \epsilon_{ij} + A_4 \epsilon_{im} \epsilon_{mj} + A_5 d_{ij} + A_6 d_{im} d_{mj} \\
& + A_7 (\dot{\sigma}_{im} \epsilon_{mj} + \epsilon_{im} \dot{\sigma}_{mj}) + A_8 (\dot{\sigma}_{im} d_{mj} + d_{im} \dot{\sigma}_{mj}) + A_9 \dot{\sigma}_{im} \epsilon_{mn} \dot{\sigma}_{nj} \\
& + A_{10} \epsilon_{im} \dot{\sigma}_{mn} \epsilon_{nj} + A_{11} \dot{\sigma}_{im} d_{mn} \dot{\sigma}_{nj} + A_{12} d_{im} \dot{\sigma}_{mn} d_{nj}
\end{aligned} \tag{38}$$

The response coefficients  $A_0, \dots, A_{12}$  are scalar functions of fifteen joint invariants of  $\dot{\sigma}_{pq}$ ,  $\epsilon_{mn}$ , and  $d_{rs}$ . Equation 38 reduces to the Cauchy elastic constitutive equation (equation 12) if dependence on  $\dot{\sigma}_{pq}$  and  $d_{rs}$  disappears. If dependence on  $\dot{\sigma}_{pq}$  and  $\epsilon_{mn}$  disappears, equation 38 will reduce to the constitutive equation of the Stokesian fluid (equation 31). If dependence on  $\dot{\sigma}_{pq}$  disappears, equation 38 will reduce to the constitutive equation of the nonlinear Kelvin solid (equation 35). Equation 38 is, therefore, very general, and by proper selection of the response coefficients  $A_0, \dots, A_{12}$ , it can be used to describe the mechanical behavior of many complex materials.

### Summary of Constitutive Equations

30. In summary, seven basic types of isotropic constitutive equations that can be utilized to describe the mechanical behavior of real materials have been discussed. These constitutive equations are all expressed in general functional forms. The specific forms of the functionals, for any particular material, must be determined through experimental observation or by physical postulates. The constitutive equations are:

- a. Cauchy elastic materials (equation 12)
- b. Hyperelastic materials (equation 13)
- c. Hypoelastic materials (equation 25)
- d. Hygrosteric materials (equation 29)
- e. Stokesian fluids (equation 31)
- f. Viscoelastic materials (equation 9)
- g. Viscoplastic materials (equation 38)

Equations 12 and 13 are used to describe the behavior of elastic materials (linear or nonlinear). Equations 25 and 29 are used to describe the behavior of rate-independent hysteretic (or plastic) materials. Equation 31 is used to describe the behavior of various fluids. Equation 9 is used to describe the behavior of various rate-dependent (viscoelastic) materials. Equation 38 is used to describe the behavior of rate-dependent hysteretic (viscoplastic) materials.

31. In the case of hysteretic materials, the constitutive equations must be accompanied by a set of criteria (or logics) defining initial loading, unloading, and subsequent reloading of the material. These criteria must be stated in terms of the invariants of the stress and/or strain tensors in order to remain invariant with respect to rigid motion of spatial coordinates. The hysteretic behavior of the material is then taken into account by using different sets of constitutive equations for loading, unloading, and reloading. There are several criteria for defining loading, unloading, and reloading that are presently being used in the solution of two-dimensional boundary-value problems for hysteretic materials. These criteria are not necessarily conclusive and although they yield satisfactory results for certain well-defined stress paths, they might lead to unrealistic material behavior under more complex loading conditions. A summary of these criteria is given in the following section.

#### Criteria for Initial Loading, Unloading, and Reloading

32. One common criterion for differentiating among initial loading, unloading, and reloading is based on the time-rate (or increment) of the first invariant of the stress tensor  $\dot{\sigma}_{ss}$ . According to this criterion,  $\dot{\sigma}_{ss} > 0$  defines loading and  $\dot{\sigma}_{ss} < 0$  defines unloading. Whenever  $\dot{\sigma}_{ss} > 0$  but  $\sigma_{ss}$  is less than its previous maximum value, the material is assumed to be reloading. A similar criterion based on the time-rate of the first invariant of the strain tensor  $\dot{\epsilon}_{ss}$  can also be used to define various loading conditions. Both of these criteria, however, will lead to unrealistic material behavior during a load-unload

cycle where the medium simultaneously experiences loading in shear and unloading in pressure or vice versa.

33. Another criterion is based on the time-rate of the octahedral shearing stress  $\dot{\tau}_{oct}$ . According to this criterion,  $\dot{\tau}_{oct} > 0$  defines loading and  $\dot{\tau}_{oct} < 0$  defines unloading. The material is assumed to be reloading whenever  $\dot{\tau}_{oct} > 0$  and  $\tau_{oct}$  is less than its previous maximum value. The condition  $\dot{\tau}_{oct} = 0$  is referred to as a neutral state of loading. The  $\dot{\tau}_{oct}$  criterion is less controversial than the  $\dot{\sigma}_{ss}$  or  $\dot{\epsilon}_{ss}$  criterion.

34. In an attempt to overcome some of the theoretical and experimental problems that can arise through the use of the  $\dot{\tau}_{oct}$  or  $\dot{\sigma}_{ss}$  criterion, a combination of the two is often utilized that separates the deformation into the hydrostatic and the deviatoric components. The  $\dot{\sigma}_{ss}$  criterion is used for the hydrostatic part of the deformation, and the  $\dot{\tau}_{oct}$  criterion is used for the deviatoric components. In this manner, it is possible for the material to unload in shear while loading in pressure or vice versa.

35. The rate of work  $\sigma_{ij}\dot{\epsilon}_{ij}$  is also used as a criterion for defining various loading conditions in a cyclic test. According to this criterion,  $\sigma_{ij}\dot{\epsilon}_{ij} > 0$  defines loading and reloading;  $\sigma_{ij}\dot{\epsilon}_{ij} < 0$  defines unloading; and the condition  $\sigma_{ij}\dot{\epsilon}_{ij} = 0$  is referred to as a neutral state of loading. The rate of work criterion and the  $\dot{\tau}_{oct}$  criterion are used extensively in the theory of plasticity, and they are essentially the same if the constitutive equation associated with them is linear from the tensorial point of view.

36. The neutral states of loading associated with the  $\dot{\tau}_{oct}$  or  $\sigma_{ij}\dot{\epsilon}_{ij}$  criterion impose certain restrictions on the material constants in the constitutive equations for loading and unloading and require special considerations. The material constants must be chosen so that the loading and unloading constitutive equations become identical whenever  $\dot{\tau}_{oct} = 0$  or  $\sigma_{ij}\dot{\epsilon}_{ij} = 0$ , i.e. neutral loading. This requirement must be met in order to obtain a unique solution for a given boundary-value problem involving cyclic loading conditions.

PART IV: SUMMARY OF DAMPING MODELS USED IN  
ENGINEERING ANALYSES OF VIBRATORY SYSTEMS

37. Most of the damping models used in engineering analyses of vibratory systems stem from one-dimensional representation of the isotropic constitutive equations presented in Part III and, therefore, are limited in their application. The models can be divided into two general categories: viscous damping models and hysteretic damping models. Various forms of these models are discussed in the following sections and are given in their three-dimensional representation for a broader application and utilization. Appendix A gives a bibliography of source material related to the model category and form.

Viscous Damping Models

38. The viscous damping models are frequently represented in terms of rheological models consisting of linear springs and viscous elements or dashpots. The viscous elements account for the energy-dissipation properties of the material, while the springs represent the elastic behavior of the material. Three types of such models are commonly used in engineering: Kelvin-Voigt, Maxwell, and standard-linear solid models. The Kelvin-Voigt model has been used more extensively due to its mathematical simplicity. The three models are presented in the order of increasing complexity.

Kelvin-Voigt model

39. The rheological model of Kelvin-Voigt material consists of a linear spring and a dashpot in parallel and corresponds to a one-degree-of-freedom damped system in the theory of vibrations. The constitutive equation of Kelvin-Voigt material is given by

$$\sigma_{ij} = (\lambda I_1 + \lambda_v \dot{I}_1) \delta_{ij} + 2\mu \epsilon_{ij} + 2\mu_v \dot{\epsilon}_{ij} \quad (39)$$

Equation 39 involves four material constants,  $\lambda$ ,  $\lambda_v$ ,  $\mu$ , and  $\mu_v$ , that must be determined experimentally. The one-dimensional

representation of equation 39, which is most commonly used in engineering application, can be obtained by allowing  $\epsilon_{12} = \epsilon_{13} = \epsilon_{23} = \epsilon_{22} = \epsilon_{33} = d_{12} = d_{13} = d_{23} = d_{22} = d_{33} = 0$ . Accordingly, in view of equations 5, 6, and 10, the one-dimensional representation of equation 39 becomes

$$\sigma_{11} = \lambda \epsilon_{11} + \lambda_v \frac{d\epsilon_{11}}{dt} + 2\mu \epsilon_{11} + 2\mu_v \frac{d\epsilon_{11}}{dt} \quad (40a)$$

$$\sigma_{22} = \lambda \epsilon_{11} + \lambda_v \frac{d\epsilon_{11}}{dt} \quad (40b)$$

where  $\epsilon_{11}$  is the strain in the direction of motion in a uniaxial strain configuration and  $\sigma_{22}$  is the lateral stress required to prevent lateral strain. Equation 40a can be written in a more compact form by collecting terms, i.e.

$$\sigma_{11} = (\lambda + 2\mu) \epsilon_{11} + (\lambda_v + 2\mu_v) \frac{d\epsilon_{11}}{dt} \quad (41)$$

The terms  $\lambda + 2\mu$  and  $\lambda_v + 2\mu_v$  correspond to the elastic modulus  $M$  and the distributed viscosity  $\eta$ , respectively, associated with the conditions of uniaxial strain. In view of the above terminology, equation 41 becomes

$$\sigma_{11} = M \epsilon_{11} + \eta \frac{d\epsilon_{11}}{dt} \quad (42)$$

Equation 42 is the counterpart of the differential equation of motion for the free vibration of a one-degree-of-freedom damped system with equivalent spring constant  $k_{eq}$  and damping coefficient  $c_{eq}$  given by

$$k_{eq} = \frac{M\bar{a}}{h} \quad (43a)$$

$$c_{eq} = \frac{\eta\bar{a}}{h} \quad (43b)$$

where  $\bar{a}$  and  $h$  are area and length, respectively. In practice,  $\bar{a}$  corresponds to the cross-sectional area of a continuous system



(a column) and  $h$  corresponds to the grid spacing of a discrete system (a lumped-parameter model) that is mathematically equivalent to the continuous system.

40. Various nonlinear forms of Kelvin-Voigt constitutive equations are also available that can be used to describe the stress-strain-time behavior of soil under various states of stress and deformation. The basic forms of these models are the same as equation 39. The constitutive coefficients,  $\lambda$ ,  $\lambda_v$ ,  $\mu$ , and  $\mu_v$ , however, are no longer constant. One of the more general nonlinear versions of the Kelvin-Voigt constitutive equation that is particularly suited for soil has the following representation:<sup>9</sup>

$$\begin{aligned} \sigma_{ij} = & \left\{ (\sigma_0 + \sigma_B) [\exp(aI_1)] - \sigma_B \right\} \delta_{ij} \\ & + \frac{b_0 \left\{ (\sigma_0 + \sigma_B) [\exp(aI_1)] - \sigma_B \right\}^{b_1} \left( \epsilon_{ij} - \frac{I_1}{3} \delta_{ij} \right)}{1 + \frac{b_0}{3\bar{\alpha}} (3I_2 - I_1^2)^{1/2} \left\{ (\sigma_0 + \sigma_B) [\exp(aI_1)] - \sigma_B \right\}^{b_1-1}} \\ & + \frac{\eta_c \left\{ \ln \left[ 1 + \frac{1}{3} (3\dot{I}_2 - \dot{I}_1^2)^{1/2} \right] \right\}^n \left( \dot{\epsilon}_{ij} - \frac{\dot{I}_1}{3} \delta_{ij} \right)}{\frac{1}{3} (3\dot{I}_2 - \dot{I}_1^2)^{1/2}} \end{aligned} \quad (44)$$

where  $\sigma_0$ ,  $\sigma_B$ ,  $a$ ,  $b_0$ ,  $b_1$ ,  $\bar{\alpha}$ ,  $\eta_c$ , and  $n$  are material constants that must be determined experimentally. Equation 44 was derived for cohesionless materials and includes the following material characteristics:

- a. Nonlinear pressure-volume behavior
- b. Nonlinear shear stress-shear deformation behavior
- c. Effect of pressure on shear resistance
- d. Plastic failure
- e. Nonlinear viscous effects

A similar constitutive equation was also formulated for cohesive materials.<sup>9</sup>

#### Maxwell model

41. The rheological model of the Maxwell material consists of a linear spring and a dashpot connected in series. The constitutive equation of Maxwell material is given by

$$\dot{\sigma}_{ij} = (\alpha_m J_1 + \lambda_m \dot{I}_1) \delta_{ij} + 2\beta_m \sigma_{ij} + 2\mu_m d_{ij} \quad (45)$$

where  $\alpha_m$ ,  $\lambda_m$ ,  $\beta_m$ , and  $\mu_m$  are material constants that must be determined experimentally. The one-dimensional representation of equation 45 can be obtained by setting  $d_{12} = d_{13} = d_{23} = d_{22} = d_{33} = 0$ . Accordingly, in view of equations 5, 6, and 10, the one-dimensional representation of equation 45 becomes

$$\dot{\sigma}_{11} = \alpha_m J_1 + \lambda_m \frac{d\epsilon_{11}}{dt} + 2\beta_m \sigma_{11} + 2\mu_m \frac{d\epsilon_{11}}{dt} \quad (46a)$$

$$\dot{\sigma}_{22} = \alpha_m J_1 + \lambda_m \frac{d\epsilon_{11}}{dt} + 2\beta_m \sigma_{22} \quad (46b)$$

where, as before,  $\epsilon_{11}$  is the strain in the direction of motion in uniaxial strain configuration and  $\sigma_{22}$  is the lateral stress required to prevent lateral strain. Equations 46a and 46b describe the motion of a Maxwell material in a uniaxial strain condition. These equations are coupled differential equations and must be integrated in order to obtain stress-strain-time relations for the material.

#### Standard-linear solid

42. The rheological model of the standard-linear solid consists of a Kelvin-Voigt element and a linear spring connected in series. The constitutive equation of the standard-linear solid may be expressed as

$$\dot{\sigma}_{ij} = (\lambda_s I_1 + \bar{\lambda}_s \dot{I}_1 + \alpha_s J_1) \delta_{ij} + 2\beta_s \sigma_{ij} + 2\bar{\mu}_s d_{ij} + 2\mu_s \epsilon_{ij} \quad (47)$$

where  $\lambda_s$ ,  $\bar{\lambda}_s$ ,  $\alpha_s$ ,  $\bar{\mu}_s$ , and  $\mu_s$  are material constants that must be

determined experimentally. Equation 47 can also be expressed in one-dimensional form similar to the Kelvin-Voigt and Maxwell models.

#### Combined models

43. Various other combinations of spring and dashpot models can be constructed by combining, for example, Kelvin-Voigt elements with Maxwell elements in series. It should be pointed out, however, that the Kelvin-Voigt, Maxwell, and standard-linear solid models, or other combinations of these models, are special forms of equations 9 and 30

#### Hysteretic Damping Models

44. The hysteretic damping models are generally formulated in terms of isotropic incremental elastic-plastic constitutive equations. The nonlinear behavior is incorporated into such constitutive equations by making the elastic moduli functions of the strain and/or stress invariants. Hysteretic behavior is taken into account by using different values of moduli for loading and unloading. Classical plasticity is often incorporated into the model by specifying a yield criterion that effectively serves to limit the maximum stress deviators in the material. As mentioned previously, a set of logics or criteria must also be specified for the hysteretic damping models to define various loading conditions. Several forms of hysteretic damping models are presently being used in stress analysis studies of hysteretic materials. As mentioned above, most of these models have evolved along the lines of isotropic incremental elastic-plastic constitutive equations. A brief discussion of the mathematical formulation of isotropic models is given in the following paragraph.

#### Isotropic incremental elastic-plastic models

45. In the incremental theory of plasticity, the incremental strain tensor  $d\epsilon_{ij}$  is assumed to be composed of the plastic strain increment tensor  $d\epsilon_{ij}^p$  and the elastic strain increment tensor  $d\epsilon_{ij}^e$ , i.e.

$$d\epsilon_{ij} = d\epsilon_{ij}^p + d\epsilon_{ij}^e \quad (48)$$

The basic postulate of the plasticity theory is the existence of a yield function or yield criterion  $F$ , so that the material yields when

$$F(\sigma_{ij}) = 0 \quad (49)$$

In the absence of any further constraint, the material would then flow plastically, undergoing plastic as well as elastic strains. When the stresses are less than those required to satisfy equation 49, the material will undergo elastic strains only, i.e., the material behaves as a linear elastic solid. No stress state that exceeds the yield criterion is allowed. These conditions are summarized as

$$\begin{aligned} d\epsilon_{ij} &= d\epsilon_{ij}^e & \text{if } F < 0 \\ d\epsilon_{ij} &= d\epsilon_{ij}^e + d\epsilon_{ij}^p & \text{if } F = 0 \end{aligned} \quad (50)$$

$F > 0$  not admissible

46. The elastic relation of stress and strain was stated previously in terms of the strain energy functions  $U$  and  $\psi$  (equations 16 and 20). In terms of stress and strain, the elastic relation becomes

$$\epsilon_{ij} = \frac{1+\nu}{E} \sigma_{ij} - \frac{\nu}{E} \sigma_{kk} \delta_{ij} \quad (51)$$

from which one obtains the elastic strain increment tensor

$$d\epsilon_{ij}^e = \frac{1+\nu}{E} d\sigma_{ij} - \frac{\nu}{E} d\sigma_{kk} \delta_{ij} \quad (52)$$

The plastic strain increment tensor is derived from the yield function (equation 49) by utilizing the concept of the plastic potential,<sup>10</sup> i.e.

$$d\epsilon_{ij}^p = \Lambda \frac{\partial F}{\partial \sigma_{ij}} \quad (53)$$

in which  $\Lambda$  is a positive factor of proportionality that must be determined from the condition that the stresses must not violate equation 49. Consequently,  $\Lambda$  is not unique and has different forms for various yield functions. In view of equations 52, 53, and 48, the total strain increment tensor takes the following form

$$d\epsilon_{ij} = \frac{1+\nu}{E} d\sigma_{ij} - \frac{\nu}{E} d\sigma_{kk} \delta_{ij} + \Lambda \frac{\partial F}{\partial \sigma_{ij}} \quad (54)$$

Equation 54, in conjunction with the conditions given by equation 50, constitutes the incremental stress-strain relation of the theory of plasticity.

47. The most widely used incremental elastic-plastic model in engineering is the Prandtl-Reuss constitutive equation, which utilizes the well-known von Mises yield condition given by

$$\sqrt{\frac{1}{2} S_{mn} S_{mn}} = k \quad (55)$$

where

$S_{mn}$  = components of stress deviator tensor

$k$  = material constant

48. In view of equation 55, the yield function  $F$  for Prandtl-Reuss material becomes

$$F = \frac{1}{2} S_{mn} S_{mn} - k^2 \quad (56)$$

Employing equation 56 in equation 53, the plastic strain increment tensor becomes

$$d\epsilon_{ij}^p = \Lambda S_{ij} \quad (57)$$

From equation 57, it follows that

$$d\epsilon_{ii}^p = \Lambda S_{ii} = 0 \quad (58)$$

indicating that no plastic volume change can occur in the plastic range for the Prandtl-Reuss material. The proportionality factor  $\Lambda$  for the Prandtl-Reuss material is given by

$$\Lambda = \frac{\dot{W}}{2k^2} \quad (59)$$

where  $\dot{W}$  is the rate of work done in shear or the increment of internal energy due to distortion.<sup>11</sup> Accordingly, in view of equations 59, 57, and 54, the strain increment tensor becomes

$$d\epsilon_{ij} = \frac{1+\nu}{E} d\sigma_{ij} - \frac{\nu}{E} d\sigma_{kk} \delta_{ij} + \frac{\dot{W}}{2k^2} S_{ij} \quad (60)$$

Equation 60 is the Prandtl-Reuss constitutive equation; it applies in the plastic range when  $\dot{W} > 0$  and equation 55 holds. If  $\dot{W} < 0$ , unloading is taking place and the behavior of the material is governed by the elastic constitutive equation in the incremental form, i.e. equation 52. If  $\dot{W} = 0$ , the loading is said to be neutral, and equations 52 and 60 become identical. The above results can be summarized in the following form:

$$\left\{ \begin{array}{ll} d\epsilon_{ij} = \frac{1+\nu}{E} d\sigma_{ij} - \frac{\nu}{E} d\sigma_{kk} \delta_{ij} & \text{when } \left[ \begin{array}{l} \sqrt{\frac{1}{2} S_{mn} S_{mn}} < k \\ \text{elastic loading} \end{array} \right. \quad (61a) \\ \\ d\epsilon_{ij} = \frac{1+\nu}{E} d\sigma_{ij} - \frac{\nu}{E} d\sigma_{kk} \delta_{ij} + \frac{\dot{W}}{2k^2} S_{ij} & \text{when } \left[ \begin{array}{l} \sqrt{\frac{1}{2} S_{mn} S_{mn}} = k \\ \dot{W} > 0 \end{array} \right. \quad (61b) \\ \\ d\epsilon_{ij} = \frac{1+\nu}{E} d\sigma_{ij} - \frac{\nu}{E} d\sigma_{kk} \delta_{ij} & \text{when } \left[ \begin{array}{l} \dot{W} < 0 \\ \text{elastic unloading} \end{array} \right. \quad (61c) \end{array} \right.$$

From equation 61, it is apparent that the Prandtl-Reuss material exhibits energy dissipation in shear. However, in view of equation 58, the material behaves elastically during hydrostatic deformation.

49. Other forms of elastic-plastic constitutive equations are

also available that are more general than the Prandtl-Reuss material. For example, by utilizing the Drucker-Prager yield condition

$$\alpha \sigma_{kk} + \sqrt{\frac{1}{2} S_{mn} S_{mn}} = k \quad (62)$$

in equation 53, and utilizing the  $\dot{W}$  criterion, a more general constitutive equation can be formulated that can account for energy dissipation in shear which is accompanied by volume expansion. It is also possible to formulate a work-hardening plastic model by constructing a yield function that is dependent on the plastic strain tensor, i.e.

$$F(\sigma_{ij}, d\epsilon_{ij}^p) = 0 \quad (63)$$

The yield function then changes or moves as plastic strain takes place in order to account for the work-hardening behavior of the material.

#### Variable moduli models

50. The elastic-plastic models fail to reproduce the stress-strain behavior of highly hysteretic materials such as soil under complex and transient states of stress and, therefore, are limited in their application. In an attempt to overcome this deficiency, an alternate approach was developed that has led to the formulation of a series of constitutive models known as variable moduli models.<sup>12</sup> Unlike the elastic-plastic models, there is no explicit yield function or flow rule associated with the variable moduli models.

51. The basic constitutive relation of the variable moduli model is given by

$$d\sigma_{ij} = K d\epsilon_{kk} \delta_{ij} + 2\mu \left( d\epsilon_{ij} - \frac{1}{3} d\epsilon_{kk} \delta_{ij} \right) \quad (64)$$

where  $d$  = an incremental change in stress or strain and  $K$  = the bulk modulus. Equation 64 can also be written in terms of the deviatoric and spherical components of deformation, i.e.

$$dS_{ij} = 2\mu \left( d\epsilon_{ij} - \frac{1}{3} d\epsilon_{kk} \delta_{ij} \right) \quad (65a)$$

$$d\sigma_{kk} = 3Kd\epsilon_{kk} \quad (65b)$$

Nonlinear behavior can be incorporated in the model by making  $K$  and  $\mu$  functions of strain and/or stress invariants. The hysteretic behavior of the material is taken into account by using different values of  $K$  and  $\mu$  on loading and unloading.

52. To date, in constructing various variable moduli models, it has been assumed that volumetric strains are caused only by changes in pressure, i.e., that there is no coupling between volumetric strain and the components of the deviatoric stress tensor. Consequently, the bulk modulus is only related to pressure or the volumetric strain and is independent of stress deviations.

53. From equation 64 (or equation 65), it is apparent that the constitutive equations of variable moduli models are a set of first-order differential equations. The integration of the differential equations, for a given stress path and initial conditions, leads to stress-strain relations. Therefore, a relation between stress and strain is the outcome of the theory, but it depends completely on the stress path used to reach the final state as well as the initial state of the material.

54. The most recent version of a variable moduli model was developed by Nelson.<sup>12</sup> The load/unload criterion associated with the model is composed of the  $\dot{\epsilon}_{oct}$  criterion for the deviatoric part of deformation and  $\dot{\epsilon}_{kk}$  for the hydrostatic part.

55. For initial loading, the bulk modulus takes the form

$$K = K_{LD} = K_0 + K_1 (\epsilon_{kk}) + K_2 (\epsilon_{kk})^2 \quad (66)$$

and for unloading and reloading

$$K = K_{UN/RE} = K_{Ou} + K_{lu}p \quad (67)$$

where LD = loading, UN = unloading, RE = reloading and  $p$  = pressure. The parameters  $K_0$ ,  $K_1$ ,  $K_2$ ,  $K_{Ou}$ , and  $K_{lu}$  are



material constants that must be determined experimentally. The unloading/reloading bulk modulus is used whenever the pressure is decreasing or whenever pressure is increasing, but its magnitude is less than the maximum previous pressure.

56. The expressions for shear modulus consist of two parts. At low pressure levels, i.e. for  $p$  less than some critical pressure  $p_c$ ,

$$\mu = \begin{cases} \mu_{LD} = \mu_0 + \bar{\alpha}_1 \sqrt{\frac{1}{2} S_{mn} S_{mn}} + \alpha_1 p + \alpha_2 p^2 & (68a) \\ \mu_{UN} = \mu_{0u} + \bar{\alpha}_{1u} \sqrt{\frac{1}{2} S_{mn} S_{mn}} + \alpha_{1u} p + \alpha_{2u} p^2 & (68b) \end{cases}$$

For  $p \geq p_c$ , the shear modulus becomes independent of pressure and is a function of  $\sqrt{\frac{1}{2} S_{mn} S_{mn}}$  only. Accordingly, in view of equation 68

$$p_c = -\frac{\alpha_1}{2\alpha_2} = -\frac{\alpha_{1u}}{\alpha_{2u}} \quad (69)$$

and for  $p \geq p_c$ , the expressions for shear modulus become

$$\mu = \begin{cases} \mu_{LD} = \mu_0 - \frac{1}{4} \frac{\alpha_1^2}{\alpha_2} + \bar{\alpha}_1 \sqrt{\frac{1}{2} S_{mn} S_{mn}} & (70a) \\ \mu_{UN} = \mu_{0u} - \frac{1}{4} \frac{\alpha_{1u}^2}{\alpha_{2u}} + \bar{\alpha}_{1u} \sqrt{\frac{1}{2} S_{mn} S_{mn}} & (70b) \end{cases}$$

As indicated by equations 69 and 70, the expressions for  $\mu_{LD}$  and  $\mu_{UN}$  are continuous at  $p = p_c$ . The parameters  $\mu_0$ ,  $\bar{\alpha}_1$ ,  $\alpha_1$ ,  $\alpha_2$ ,  $\mu_{0u}$ ,  $\bar{\alpha}_{1u}$ ,  $\alpha_{1u}$ , and  $\alpha_{2u}$  are material constants that must be determined experimentally. The loading shear modulus is used whenever  $\dot{\tau}_{oct} > 0$  during both initial loading in shear and subsequent reloading in shear. The unloading shear modulus is used whenever  $\dot{\tau}_{oct} \leq 0$ . The model, therefore, makes no distinction between initial loading and subsequent reloading in shear. This logic, however, results in excessive strains during cyclic loading. To overcome this deficiency, one possibility would be to use the unloading shear modulus whenever  $\dot{\tau}_{oct} \leq 0$  or whenever  $\dot{\tau}_{oct} > 0$  but the magnitude of  $\tau_{oct}$  is less

than its previous maximum value. This logic is analogous to the logic used for bulk modulus. Another possibility is to use a linear combination of  $\mu_{LD}$  and  $\mu_{UN}$  for reloading in shear. The reloading problem, in general, is not well defined, and additional cyclic data or theoretical considerations are necessary to explain this phenomenon.

57. The variable moduli models dissipate energy under both the deviatoric and the hydrostatic stress conditions and have the capability of matching experimental data quantitatively. There is, however, a basic theoretical objection for their utilization in the solution and formulation of boundary-value problems for hysteretic materials under cyclic loading conditions. This objection stems from the load/unload criterion for shear associated with these models and the neutral loading conditions defined by  $\dot{\tau}_{oct} = 0$ . As was pointed out previously, whenever  $\dot{\tau}_{oct} = 0$ , the constitutive relations for initial loading and unloading must become identical in order to have unique solutions for the boundary-value problems of interest. From equation 68, it is obvious that the expressions for shear modulus do not become identical whenever  $\dot{\tau}_{oct} = 0$  and one can use either  $\mu_{LD}$  or  $\mu_{UN}$  at such a load/unload interface. This theoretical objection for the use of variable moduli models has led to the development of a series of elastic-plastic work-hardening constitutive models that satisfy the continuity conditions associated with neutral loading and at the same time match the experimental data quantitatively.<sup>13</sup>

58. The constitutive equations of Prandtl-Reuss material and the variable moduli models are only special forms of the hypoelastic constitutive equation (equation 25).<sup>14</sup>

## PART V: DISCUSSION AND RECOMMENDATIONS

### Discussion

59. The hysteretic character of the stress-strain behavior of earth materials can be described by phenomenological constitutive equations using two alternate procedures. In the first procedure, the hysteretic behavior of earth materials is accounted for by using different sets of constitutive relations for loading and unloading, in conjunction with a set of criteria that define various loading conditions, i.e. initial loading, unloading, and reloading. The constitutive equations in this case are time-independent. In the second procedure, a single constitutive equation is specified and the energy-dissipation properties of earth materials are accounted for by the appearance of the time derivatives in the constitutive equation, which in this case is time-dependent.

60. Either of the above-mentioned procedures can be used to solve soil dynamics boundary-value problems and will yield stress-strain curves that exhibit energy-absorbing hysteretic behavior. The mechanisms of energy dissipation in the two procedures, however, are different. Additional experimental dynamic data and theoretical studies in the areas of nonlinear viscoelasticity and plasticity are required in order to determine which of the two procedures is more appropriate for describing the stress-strain behavior of earth materials. At the present time, for reasons of practicability and depending on the type of problem being considered, some investigators use the first procedure while others prefer the second. A subsequent report (Report 2) will review the current laboratory methods of determining the damping capacity of soil; it will describe specific techniques used to evaluate hysteretic effects by both the time-independent and the time-dependent approaches.

### Recommendations

61. Research efforts presently under way at the U. S. Army

Engineer Waterways Experiment Station and elsewhere are aimed at developing material models that exhibit energy-absorbing hysteretic behavior for use as input to ground motion calculation computer codes. The technique used for formulating the constitutive equations follows the procedure for constructing hysteretic damping models. The constitutive equations are, therefore, time-independent and do not manifest viscosity effects. Very little effort has been expended in recent years to formulate mathematical material models that account for the energy-dissipation properties of the earth materials through viscosity mechanisms. These types of time-dependent material models are highly desirable for the solution of boundary-value problems involving steady-state cyclic loading conditions. Moreover, they have the added advantage of being able to account for strain-rate effects.

62. Thus, it is recommended that research efforts to develop time-dependent constitutive equations for earth materials for the formulation and solution of soil dynamics boundary-value problems continue. Such relations should be capable of qualitatively and quantitatively matching the salient nonlinear and hysteretic response characteristics of earth materials, not only as determined in the one-dimensional configuration but also under a variety of other laboratory test-boundary conditions. In the author's opinion, such efforts should be governed by the following criteria:

- a. The constitutive equations must take into account nonlinear pressure-volume behavior, nonlinear shear stress-shear deformation behavior, effects of hydrostatic stress on shear deformation, shear fracture, spalling, and plastic failure of earth materials.
- b. The numerical values of the various coefficients defining the constitutive models should have physical significance in terms of compressibility, shear strength, etc., so that when extrapolating data for the different materials, rational engineering judgments can be made as to the relative magnitudes of the constitutive coefficients based on geologic descriptions, mechanical properties, and other conventional indices. The constitutive coefficients should not be merely a set of numbers generated through a trial-and-error, black box routine to fit a given set of data.

63. The constitutive models must be used to obtain analytical or closed-form solutions to special problems, e.g. one-dimensional wave propagation, in order to study the dissipation characteristics of the models as well as the theoretical implications of the new phenomena they present, such as the requirements imposed by uniqueness and continuity considerations.

64. The nonlinear Kelvin-Voigt model discussed by Rohani (equation 44), or an extension of this model, can be used as a starting point for such an undertaking.

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## APPENDIX A: SELECTED BIBLIOGRAPHY

1. This appendix contains 83 references that are related to the subject matter presented in the main text of this report. These references are in addition to those listed as Literature Cited following the main text and are given in alphabetical order. For easy reference and informational purposes, the references are cataloged in table A1 in accordance with their relation to three basic material models discussed in the report: viscoelasticity, plasticity, and hypoelasticity.

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Table A1  
A Catalog of the Selected Bibliography

Viscoelasticity	Plasticity	Hypoelasticity
Adachi, 1970	Adachi, 1970	Thomas, 1957
Adler, Sawyer, and Ferry, 1949	Baron, McCormick, and Nelson, 1969	Thomas, 1956
Alfrey, 1944	Nelson, 1969	Thomas, 1955
Alfrey and Doty, 1945	Bleich, 1970	Truesdell, 1956
Baker and Kesler, 1961	Bleich and Heer, 1963	Truesdell, 1964
Biot, 1955	Bleich and Matthews, 1968	Truesdell and Noll, 1964
Biot, 1956	Bleich and Matthews, 1967	
Biot, 1954	Bleich and Matthews, 1967	
Bland, 1960	Bleich, Matthews, and Wright, 1968	
Bland and Lee, 1955	Bleich, Matthews, and Wright, 1965	
Bodner, 1965	Bridgman, 1949	
Coleman, 1964	Brown and Swanson, 1970	
Cristescu, 1967	Christian, 1966	
Dewitt, 1955	Cristescu, 1967	
Finnie and Heller, 1959	Drucker, 1959	
Freudenthal, 1950	Drucker, 1950	
Fung, 1965	Drucker, 1956	
Gross, 1953	Drucker, 1964	
Halpin, 1968	Drucker, 1966	
Huang and Lee, 1966	Drucker, 1958	
Kondner and Krizek, 1962	Drucker and Prager, 1952	
Kondner and Krizek, 1963	Drucker and White, 1950	
Lee, 1955	Drucker, Gibson, and Henkel, 1957	
Lee and Radok, 1957	Freudenthal, 1950	
Malvern, 1969	Freudenthal and Geiringer, 1958	
Perzyna, 1963	Fung, 1965	
Prager, 1961	Green and Naghdi, 1965	
Radok, 1957	Hill, 1958	
Reed, 1950	Hill, 1950	
Reiner, 1960	Il'yushin, 1961	
Reiner, 1949	Jenike and Shield, 1959	
Reiner and Abir, 1964	Malvern, 1969	
Rivlin, 1965	Matthews and Bleich, 1969	
Schapery, 1966	Nelson, 1970	
Schapery, 1962	Nelson and Baron, 1968	
Schiffman, 1959	Prager, 1959	
Schmid, Klausner and Whitmore, 1969	Prager, 1961	
Scott Blair, 1945	Prager, 1945	
Stroganov, 1964	Reiner, 1960	
Symonds, 1965	Stroganov, 1958	
Volterra, 1951	Swanson, 1970	
	Thomas, 1961	
	Weidler and Paslay, 1969	